NATURE OF MATHEMATICS (2AB)

The universe is made up of galaxies, mountains, organisms, vehicles, and a wide variety of other things, each seemingly unique. Moreover, these things interact with one another in all sorts of ways, often violently but sometimes with great subtlety. But thanks to mathematics, people are able to think about the world of objects and events and to communicate those thoughts in ways that reveal unity and order. Mathematics is a universal language for describing patterns and relationships, both about abstractions and about objects and events in the real world. The results of theoretical and applied mathematics often influence one another, contributing to a better understanding of the world.

The map is organized around three strands—*study of patterns, universal language for describing things,* and *connections to science and technology*. They reflect key understandings students are expected to develop about mathematics as the study of patterns that involves the interplay of theory and application and that both draws on and contributes to science and technology.

In the elementary grades, the focus is on the use of quantities and shapes to describe real-world objects and events. Middle-school benchmarks emphasize the role of mathematics in representing change over time, negative quantities, and data sets. High-school benchmarks emphasize the role of mathematics in modeling real-world events and technological designs. Connections are made to benchmarks describing the contribution of computers to these mathematical activities.

In addition to other maps in this chapter, the map draws upon and contributes to maps in Chapter 9: THE MATHEMATICAL WORLD, in Chapter 11: COMMON THEMES, and in Chapter 12: HABITS OF MIND.

The use of mathematical models is a special case of how mathematics contributes to science and technology. The progression of understanding about mathematical models is laid out in the **MATHEMATICAL MODELS** map in *Atlas 1*. The use of mathematical models in science and technology is developed in greater depth in the **MODELS** map.

NOTES

Many specific ideas about the uses of mathematics (such as using shapes and numbers to describe objects and events or using graphs to help make predictions about phenomena), contribute to the 6-8 benchmark "Mathematics is helpful in almost every kind of human endeavor…." More abstract ideas about and uses of mathematics (such as those involving negative numbers, measures of central tendency, and mathematical modeling) contribute to the 9-12 benchmark "Mathematics provides a precise language…" and to a more in-depth understanding of the utility of mathematics in all fields of human endeavor, not just in science and technology.

Most mathematical theories have applications, and much work in mathematics is motivated by seeking answers to applied problems, often in science and technology. This connection is depicted by the arrows linking 9-12 benchmarks "Developments in mathematics often stimulate…" and "Developments in science or technology often stimulate…" to the 9-12 benchmark "Theories and applications in mathematical work…."

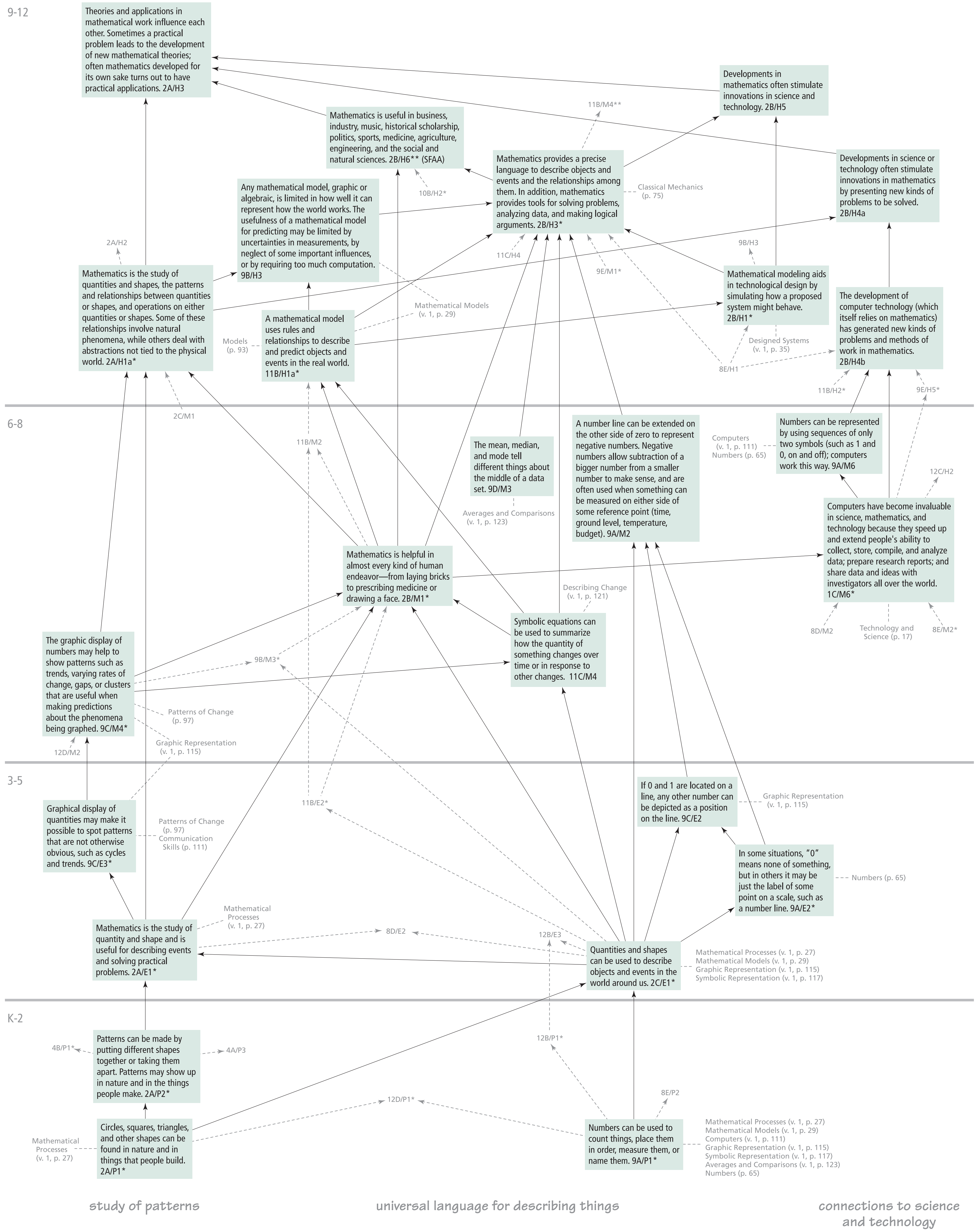
RESEARCH IN *BENCHMARKS*

Early reviews of the research (e.g., McLeod, 1992; Schoenfeld, 1992; Thompson, 1992) support continuing studies of beliefs, attitudes, and values about the nature of mathematics. For descriptions of international research programs, see the volume edited by Leder, Pehkonen, and Toerner (2002), and the special issue on affect published by *Educational Studies in Mathematics* (e.g., Zan, Brown, Evans, & Hannula, 2006). For a more comprehensive review of recent literature, with an emphasis on teacher beliefs, see Philipp (in press). Data on student beliefs and attitudes are also a component of large-scale studies, such as the National Assessment of Educational Progress and the Third International Mathematics and Science Study (see, e.g., Silver & Kenney, 2000). Investigations of students' beliefs about the nature of mathematics are illuminated by Schoenfeld (1992) and summarized in a report from the National Research Council (Kilpatrick, Swafford, & Findell, 2001). These studies examined difficulties arising from students' beliefs about the nature of mathematical problem solving and from their perceptions of the role of memorization in learning mathematics and of mathematics as rule-oriented versus process-oriented or as a static versus a dynamic discipline. There has been less emphasis on students' understanding of the relationships between mathematics, science, and technology or to the nature of mathematical inquiry as a modeling process.

Research on student understanding of early algebra ideas, such as patterns and relationships, has received increased attention. Early studies suggested students have difficulty connecting mathematical expressions, sentences, and sequences that share common structural patterns, often focusing instead on incidental similarities or differences (Ericksen, 1991). Evidence suggests that these difficulties can be ameliorated by introducing early algebra ideas in the elementary grades (Carpenter, Franke, & Levi, 2003).

Research on the integration of new technologies into mathematics instruction continues to advance and is summarized in a National Research Council report (Kilpatrick, Swafford, & Findell, 2001). Kieran and Sfard (1999) discuss how students develop stronger conceptual knowledge of equations through graphing, and there is evidence that students can use technological tools to improve problem-solving even when their algebraic skills are limited (Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000). Survey data suggest that middle- and high-school students think that mathematics has practical, everyday uses and that mathematics is more important for society than for them personally (Brown et al., 1988).

Typical student beliefs about mathematical inquiry that limit students' mathematical behavior (Schoenfeld, 1985, 2006) include the following: There is only one correct way to solve any mathematics problem and only one correct answer; mathematics is done by individuals in isolation; mathematical problems can be solved quickly or not at all and their solutions do not have to make sense; and formal proof is irrelevant to processes of discovery and invention (Schoenfeld, 1985, 1989a, 1989b). Research is needed to assess when and how students can understand that mathematical inquiry is a cycle in which ideas are represented abstractly, abstractions are manipulated, and results are tested against the original ideas. We must also learn at what age students can begin to represent something by a symbol or expression, and what standards students use to judge when solutions to mathematical problems are useful or adequate.


9-12
Theories and applications in mathematical work influence each other. Sometimes a practical problem leads to the development of new mathematical theories; often mathematics developed for its own sake turns out to have practical applications. 2A/H3
Developments in mathematics often stimulate innovations in science and technology. 2B/H5
11B/M4**
Mathematics is useful in business, industry, music, historical scholarship, politics, sports, medicine, agriculture, engineering, and the social and natural sciences. 2B/H6** (SFAA)
Mathematics provides a precise language to describe objects and events and the relationships among them. In addition, mathematics provides tools for solving problems, analyzing data, and making logical arguments. 2B/H3*
Developments in science or technology often stimulate innovations in mathematics by presenting new kinds of problems to be solved. 2B/H4a
Classical Mechanics (p. 75)
10B/H2*
Any mathematical model, graphic or algebraic, is limited in how well it can represent how the world works. The usefulness of a mathematical model for predicting may be limited by uncertainties in measurements, by neglect of some important influences, or by requiring too much computation. 9B/H3
2A/H2
11C/H4
9E/M1*
9B/H3
Mathematics is the study of quantities and shapes, the patterns and relationships between quantities or shapes, and operations on either quantities or shapes. Some of these relationships involve natural phenomena, while others deal with abstractions not tied to the physical world. 2A/H1a*
Mathematical modeling aids in technological design by simulating how a proposed system might behave. 2B/H1*
The development of computer technology (which itself relies on mathematics) has generated new kinds of problems and methods of work in mathematics. 2B/H4b
Mathematical Models (v. 1, p. 29)
A mathematical model uses rules and relationships to describe and predict objects and events in the real world. 11B/H1a*
Models (p. 93)
Designed Systems (v. 1, p. 35)
8E/H1
11B/H2*
9E/H5*
6-8
2C/M1
11B/M2
The mean, median, and mode tell different things about the middle of a data set. 9D/M3
A number line can be extended on the other side of zero to represent negative numbers. Negative numbers allow subtraction of a bigger number from a smaller number to make sense, and are often used when something can be measured on either side of some reference point (time, ground level, temperature, budget). 9A/M2
Computers (v. 1, p. 111)
Numbers (p. 65)
Numbers can be represented by using sequences of only two symbols (such as 1 and 0, on and off); computers work this way. 9A/M6
12C/H2
Averages and Comparisons (v. 1, p. 123)
Computers have become invaluable in science, mathematics, and technology because they speed up and extend people's ability to collect, store, compile, and analyze data; prepare research reports; and share data and ideas with investigators all over the world. 1C/M6*
Mathematics is helpful in almost every kind of human endeavor—from laying bricks to prescribing medicine or drawing a face. 2B/M1*
Describing Change (v. 1, p. 121)
Symbolic equations can be used to summarize how the quantity of something changes over time or in response to other changes. 11C/M4
8D/M2
Technology and Science (p. 17)
8E/M2*
The graphic display of numbers may help to show patterns such as trends, varying rates of change, gaps, or clusters that are useful when making predictions about the phenomena being graphed. 9C/M4*
9B/M3*
Patterns of Change (p. 97)
Graphic Representation (v. 1, p. 115)
12D/M2
3-5
11B/E2*
If 0 and 1 are located on a line, any other number can be depicted as a position on the line. 9C/E2
Graphic Representation (v. 1, p. 115)
Graphical display of quantities may make it possible to spot patterns that are not otherwise obvious, such as cycles and trends. 9C/E3*
Patterns of Change (p. 97)
Communication Skills (p. 111)
In some situations, "0" means none of something, but in others it may be just the label of some point on a scale, such as a number line. 9A/E2*
Numbers (p. 65)
Mathematical Processes (v. 1, p. 27)
Mathematics is the study of quantity and shape and is useful for describing events and solving practical problems. 2A/E1*
8D/E2
12B/E3
Quantities and shapes can be used to describe objects and events in the world around us. 2C/E1*
Mathematical Processes (v. 1, p. 27)
Mathematical Models (v. 1, p. 29)
Graphic Representation (v. 1, p. 115)
Symbolic Representation (v. 1, p. 117)
K-2
4B/P1*
Patterns can be made by putting different shapes together or taking them apart. Patterns may show up in nature and in the things people make. 2A/P2*
4A/P3
12B/P1*
8E/P2
12D/P1*
Mathematical Processes (v. 1, p. 27)
Circles, squares, triangles, and other shapes can be found in nature and in things that people build. 2A/P1*
Numbers can be used to count things, place them in order, measure them, or name them. 9A/P1*
Mathematical Processes (v. 1, p. 27)
Mathematical Models (v. 1, p. 29)
Computers (v. 1, p. 111)
Graphic Representation (v. 1, p. 115)
Symbolic Representation (v. 1, p. 117)
Averages and Comparisons (v. 1, p. 123)
Numbers (p. 65)
study of patterns
universal language for describing things
connections to science and technology